Tutte's three-edge-colouring conjecture

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joint with Neil Robertson and Robin Thomas

The four-colour theorem

Appel & Haken, 1977; Robertson, Sanders, S., Thomas, 1997

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The four-colour theorem

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Every loopless planar graph is four-vertex-colourable.

Equivalently (Tait, 1880):

Every planar cubic graph with no cut-edge is three-edge-colourable.

Non-planar cubic graphs

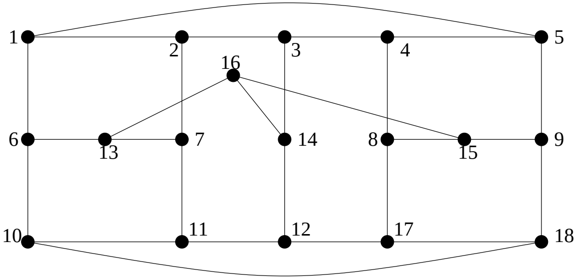
The Petersen graph is not three-edge-colourable.

Tutte' conjecture, 1966

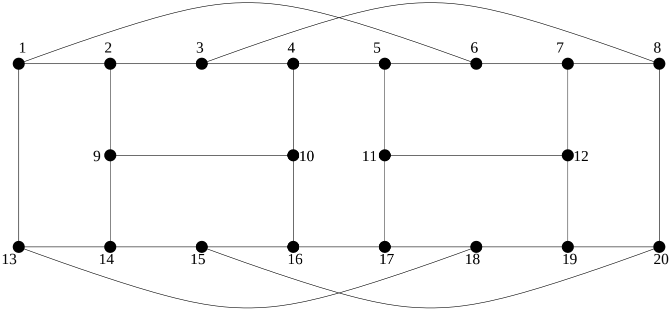
Every cubic graph with no cut-edge not containing Petersen as a minor is three-edge-colourable.

This is now a theorem.

G is apex if $G \setminus v$ is planar for some vertex v.



G is doublecross if G can be drawn in the plane with only two crossings, both on the outside.



Apex and doublecross graphs do not have Petersen minors.

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Theorem (Sanders, S., Thomas, 1997)

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Proof: Redo the proof of the four-colour theorem (twice).

G is C5C if

- G is cubic and $|V(G)| \geq 8$, and
- for every partition (X, Y) of $V(G)$ with $|X|, |Y| \geq 3$ there are at least five edges between X and Y.

G is theta-connected if

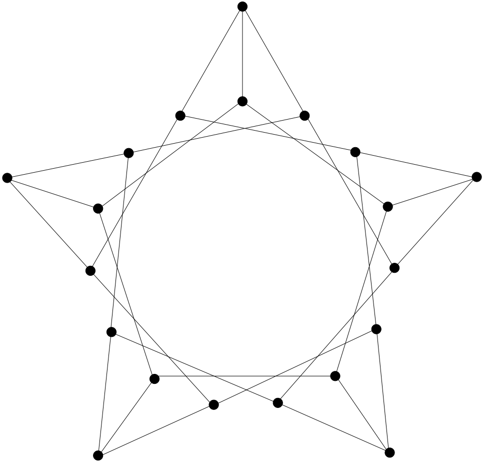
- G is C5C, and
- for every partition (X, Y) of $V(G)$ with $|X|, |Y| \geq 7$ there are at least six edges between X and Y.

Theorem (RST, 1997)

Any minimal counterexample to Tutte's conjecture is either theta-connected or apex.

Theorem (Main topic of this talk)

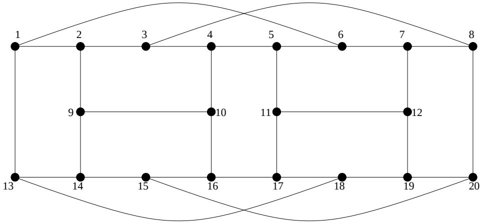
Every theta-connected graph not containing Petersen is either apex or doublecross, except Starfish.



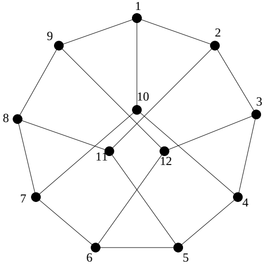
Theorem

Let G be theta-connected, and not contain Petersen. If G

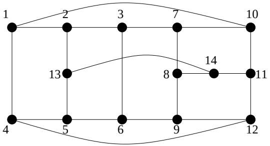
- *contains Starfish then G is Starfish*
- *contains Jaws then G is doublecross*
- *contains neither of Jaws and Starfish then G is apex.*



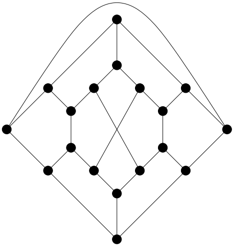
Four important graphs



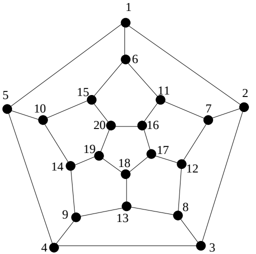
Triplex



Box



Ruby



Dodecahedron

Theorem

- *Petersen, Triplex, Box, Ruby and Dodecahedron are the only minimal graphs that are C5C (McCuaig, 1992)*

Theorem

- *Petersen, Triplex, Box, Ruby and Dodecahedron are the only minimal graphs that are C5C (McCuaig, 1992)*
- *Petersen, Triplex, Box, Ruby are the only minimal graphs that are C5C and non-planar.*

G is doe-connected if

- G is C5C and
- if (X, Y) is a partition of $V(G)$ with $|X|, |Y| \geq 7$, and there are only five edges between X and Y, then contracting Y gives a non-planar graph.

G is arched if $G \setminus e$ is planar for some edge e.

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Theorem

- *Petersen, Triplex, Box are the only minimal graphs that are doe-connected and with crossing number at least two.*

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Theorem

- *Petersen, Triplex, Box are the only minimal graphs that are doe-connected and with crossing number at least two.*
- *Petersen, Triplex are the only minimal graphs that are doe-connected and not arched.*

Trying to prove: if G is theta-connected with no Petersen and no Jaws or Starfish, then G is apex. We may assume G contains Triplex.

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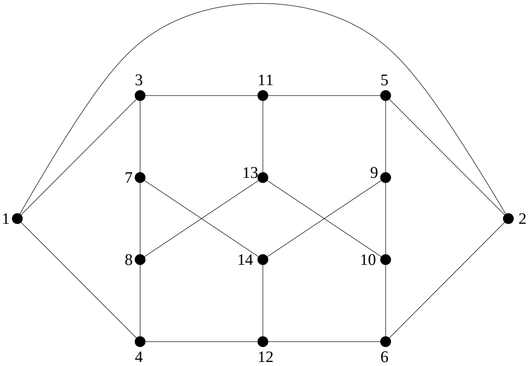
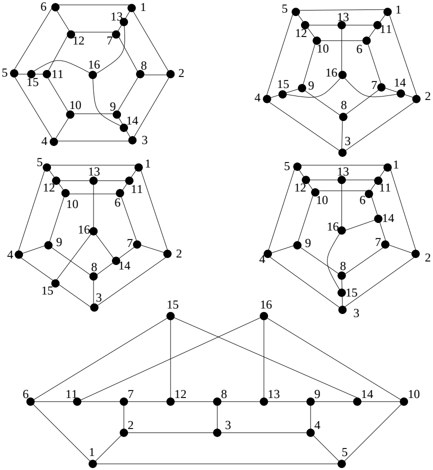
Theorem

We may assume G contains Drum.



Theorem

We may assume G contains one of Firstapex, Secondapex, Thirdapex, Fourthapex, Sailboat.



G is doubly-apex if G is cubic and identifying some two vertices makes it planar.

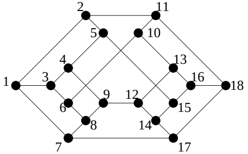
Theorem

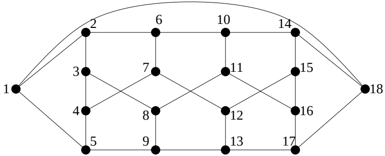
Petersen, Firstapex, Secondapex, Thirdapex, Fourthapex are the only minimal graphs that are doe-connected and not arched or doubly-apex.

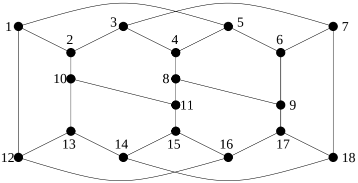
So we may assume G contains one of Firstapex, Secondapex, Thirdapex, Fourthapex.

Theorem

Petersen, Diamond, Bigdrum, Concertina are the only minimal graphs that are doe-connected and not apex.





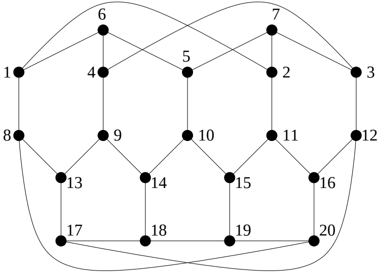


G is die-connected if

- G is C5C, and
- for every partition (X, Y) of $V(G)$ with $|X|, |Y| \geq 9$ there are at least six edges between X and Y.

Theorem

Petersen, Jaws, Starfish, and six more (below) are the only minimal graphs that are die-connected and not apex.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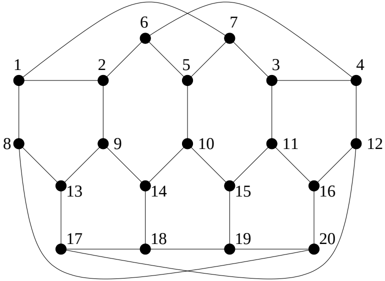
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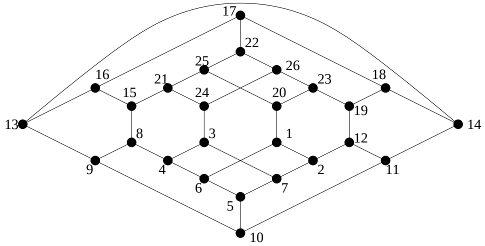
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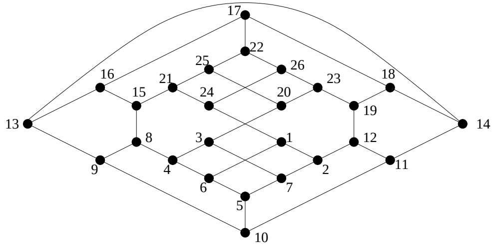
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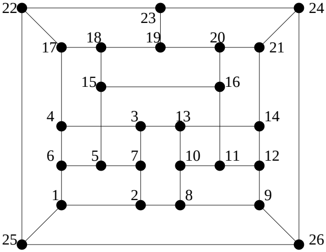
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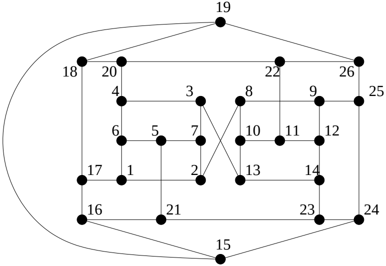
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Recall: G is theta-connected if

- G is C5C, and
- for every partition (X, Y) of $V(G)$ with $|X|, |Y| \geq 7$ there are at least six edges between X and Y.

Theorem

Petersen, Jaws, Starfish are the only minimal graphs that are theta-connected and not apex.